Weekly 1 Solutions

Overview: Below are some possible solutions to the exercises from Weekly 1. Keep in mind that there may be many ways to solve each problem. Your grade on Weekly 1 will be determined by the grader using the Weekly Assignment Rubric.

Solutions:

1. Exercise 9.1.15

Solution. a. This trace desribes how the volume V of 1 mole of an ideal gas is related to the temperature T if the pressure P is fixed at $P = 1000$ pascals. The trace is the curve described by the equation $z = V(1000, T) = 0.008314T$. Since this curve is a line, we can be very specific: the trace tells us that when the pressure is fixed at $P = 1000$ pascals, the volume V of 1 mole of a gas increases linearly with the temperature T at a rate of 0.008314 cubic meters per unit Kelvin.

b. This trace describes how the volume of 1 mole of a gas is related to the pressure P if the temperature is fixed at $T = 5$ Kelvin. The trace is the hyperbola described by the equation $z = 41.57/P$. This tells us that when the temperature is fixed at $T = 5$ Kelvin, the volume is inversely related to the pressure P. More specifically, if the temperature is fixed at $T = 5$ Kelvin, then the volume of 1 mole of a gas decreases as the pressure increases, and vice-versa

c. The level curve with $V = 0.5$ describes the relationship between the temperature T and the pressure P when the volume of 1 mole of a gas is fixed at $V = 0.5$ cubic meters. As long as the pressure is non-zero (meaning we are not in a perfect vacuum), the equation of the curve is $P = 16.628T$. This tells us that when the volume is fixed at $P = 0.5$ cubic meters, the pressure P increases linearly with the temperature T at a rate of 16.628 pascal per Kelvin.

d. You should have drawn a picture. I will just graph it: GeoGebra: Ideal Gas Law. Note that temperature are pressure are nonnegative by definition, therefore the domain of this function is $D = \{(P, T) : 0 \leq T,\ 0 \leq P\}$.

□

2. Exercise 9.1.17

Solution. a. The domain is the set of all points for which $4 - x^2 - y^2 \geq 0$, since the square root of a negative number is not a real number. More precisely, the domain D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

This is a closed disk in the xy-plane centered at the origin with radius 4: GeoGebra: A Disk.

b. Any point (x, y) in D satisfies $0 \leq x^2 + y^2 \leq 4$. Therefore, the range is the set

$$R = \{8 - \sqrt{t} \in \mathbb{R} : t \in [0, 4]\} = [6, 8].$$

c. The traces are circles, I used $c = 6.25, 6.5, 6.75, 7$ and got this: GeoGebra: Exercise 9.1.17.c.

d. I used $c = -1, 0, 1, 1.5, 1.75$ and got this: Exercise 9.1.17.d.

e. The traces are exactly the same curves (just in different directions on the graph).

f. Using just the traces, you can conclude that the graph is a circular bowl shape centered on the z axis and opening in the positive z direction. Using the range you can determine where the top and the bottom of the bowl are. Here's the graph: GeoGebra: Exercise 9.1.17 graph.

□

3. Exercise 9.2.16

Solution. a. The vectors $\mathbf{u}$ and $\mathbf{v}$ form right triangles with the horizontal. The hypotenuses of the triangles are $|\mathbf{u}|$ and $|\mathbf{v}|$, respectively. Using trigonometry, it is clear that

$$\mathbf{u} = -|\mathbf{u}|\cos(60)\mathbf{i} + |\mathbf{u}|\sin(60)\mathbf{j} = -\frac{1}{2}|\mathbf{u}|\mathbf{i} + \frac{\sqrt{3}}{2}|\mathbf{u}|\mathbf{j}$$

and

$$\mathbf{v} = |\mathbf{v}|\cos(45)\mathbf{i} + |\mathbf{v}|\sin(45)\mathbf{j} = \frac{\sqrt{2}}{2}|\mathbf{v}|\mathbf{i} + \frac{\sqrt{2}}{2}|\mathbf{v}|\mathbf{j}.$$

b. We are given that the magnitude of the force of gravity $\mathbf{w}$ acting to pull the picture downwards is 50 pounds. Since we assumed the wires meet at the origin, we can take $w = -50\mathbf{j}$.

c. The picture will hang in equilibrium if $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. Comparing components in this vector equation, we get a system of equations

$$\begin{cases} 0 &= -\frac{1}{2}|\mathbf{u}| + \frac{\sqrt{2}}{2}|\mathbf{v}| \\ 50 &= \frac{\sqrt{3}}{2}|\mathbf{u}| + \frac{\sqrt{2}}{2}|\mathbf{v}|. \end{cases}$$

Solve by elimination or substitution to get $|\mathbf{u}| = \frac{100}{\sqrt{3}+1}$ and $|\mathbf{v}| = \frac{100}{\sqrt{6}+\sqrt{2}}$. Plug back in to the expressions from part (a) to complete the solution.

□

4. Exercise 9.3.11

Solution. a. The vectors are not perpendicular because $\langle 2, -1\rangle \cdot \mathbf{v} = 2(-2) + (-1)5 = -9$.

b. Suppose that $\mathbf{u} = \langle a, b\rangle$ is a unit vector so that $1 = |\mathbf{u}|^2 = a^2 + b^2$. Further assume that $\mathbf{u}$ is perpendicular to $\mathbf{v} = \langle -2, 5\rangle$ so that $0 = \mathbf{u} \cdot \mathbf{v} = -2a + 5b$. Now, we have a system of two equations that we can solve by substitution. We have $a = \frac{5}{2}b$ so that $1 = \frac{29}{4}b^2$. The second equation has precisely two solutions, $b = \pm\frac{2}{\sqrt{29}}$. So there are precisely two vectors that fit the bill, $\mathbf{u} = \pm\langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\rangle$. This makes sense geometrically.

c. It is not, because the dot product is not zero.

d. The method to solve this part is similar to part (b). But in this case, there are infinitely many unit vectors perpendicular to $\mathbf{y}$. The set of terminal points of all possible vectors traces out a circle in $\mathbb{R}^3$ with radius equal to 1. Here's a visualization: GeoGebra: All Unit Vector Perp. to a Fixed Vector.

e. Suppose that $\mathbf{r} = \langle a, b, c\rangle$ is a unit vector perpendicular to both $\mathbf{y}$ and $\mathbf{z}$. Then we get a system of equations

$$\begin{cases} 1 &= a^2 + b^2 + c^2 \\ 0 &= 2a + b \\ 0 &= 3b - 2c. \end{cases}$$

By substituting the second two equations into the first, we get $1 =$ so that $a = \pm\frac{1}{\sqrt{14}}$. So there are two unit vectors with the desired property, namely $\mathbf{r} = \pm\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\rangle$. If you ignore the condition that $\mathbf{r}$ should have unit length, then the set of all terminal points of vectors perpendicular to both $\mathbf{y}$ and $\mathbf{z}$ is a line!
You could also have solved this part using the cross product.

□

5. (16 points) Exercise 9.4.15. Here's a useful tool for visualizing part (d): GeoGebra: Parallelepiped.

Solution. **a.** To prove this claim, just follow the definitions. We have

$$\begin{aligned}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= u_1v_2w_3 - u_1w_2v_3 - u_2v_1w_3 + u_2w_1v_3 + u_3v_1w_2 - u_3w_1v_2 \\ &= \langle u_2v_3 - u_3v_2, -(v_1u_3 - v_3u_1), u_1v_2 - u_2v_1\rangle \cdot \langle w_1, w_2, w_3\rangle \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.\end{aligned}$$

b. This follows from the anticommutativity of the cross product. We have

$$\begin{aligned}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} && \text{(by (a))} \\ &= -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} && \text{(by anticommutativity)} \\ &= -\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} && \text{(by (a)).}\end{aligned}$$

c. This follows from parts (a) and (b). We have

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} && \text{(by (a))} \\ &= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} && \text{(by b)} \\ &= (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} && \text{(by a).}\end{aligned}$$

The other claim is similar.

d. **i.** Since $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is the signed volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$, we can conclude that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \neq 0$ if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are not coplanar.

ii. If $\mathbf{u}, \mathbf{w}, \mathbf{w}$ are coplanar, then the parallelpiped has zero volume. Therefore, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

iii. It is clear that (i) and (ii) proves both implications.

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